

Relativistic mechanics in a general setting

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Abstract. Relativistic mechanics on an arbitrary manifold is formulated in the terms of jets of its one-dimensional submanifolds. A generic relativistic Lagrangian is constructed. Relativistic mechanics on a pseudo-Riemannian manifold is particularly considered.

1 Introduction

Classical non-relativistic mechanics is adequately formulated as Lagrangian and Hamiltonian theory on a fibre bundle $Q \rightarrow \mathbb{R}$ over the time axis \mathbb{R} [1, 3, 6, 8, 9, 11].

If a configuration space Q of a mechanical system has no preferable fibration $Q \rightarrow \mathbb{R}$, we obtain a general formulation of relativistic mechanics, including Special Relativity on the Minkowski space $Q = \mathbb{R}^4$ [3, 9, 11]. A velocity space of relativistic mechanics is the first order jet manifold $J_1^1 Q$ of one-dimensional submanifolds of the configuration space Q . The notion of jets of submanifolds [2, 4, 5, 7] generalizes that of jets of sections of fibre bundles which are utilized in field theory and non-relativistic mechanics (Section 2). The jet bundle $J_1^1 Q \rightarrow Q$ is projective, and one can think of its fibres as being spaces of the three-velocities of a relativistic system (Section 3). The four-velocities of a relativistic system are represented by elements of the tangent bundle TQ of the configuration space Q , while the cotangent bundle T^*Q , endowed with the canonical symplectic form, plays a role of the phase space of relativistic theory (Section 6).

We develop Lagrangian formalism on the jet bundle $J_1^1 Q \rightarrow Q$ (Section 4). We show that, in the framework of this formalism, Lagrangians possess a certain gauge symmetry (27) and, consequently, the corresponding Lagrange operators obey the rather restrictive Noether identity (28). Solving this Noether identity, we obtain the generic Lagrangian (31) and the equation of motion (51) of relativistic mechanics on a manifold Q . In particular, if Q is the Minkowski space, we are in the case of Special Relativity (Example 4).

Generalizing this example, we consider relativistic mechanics on an arbitrary pseudo-Riemannian manifold. Its equation of motion is the relativistic geodesic equation (55). Hamiltonian relativistic mechanics on a pseudo-Riemannian manifold is developed in Section 6. Its generic Hamiltonian takes the form (69).

2 Jets of submanifolds

Jets of sections of fibre bundles are particular jets of submanifolds of a manifold [2, 4, 5, 7].

Given an m -dimensional smooth real manifold Z , a k -order jet of n -dimensional submanifolds of Z at a point $z \in Z$ is defined as an equivalence class $j_z^k S$ of n -dimensional imbedded submanifolds of Z through z which are tangent to each other at z with order $k \geq 0$. Namely, two submanifolds

$$i_S : S \rightarrow Z, \quad i_{S'} : S' \rightarrow Z$$

through a point $z \in Z$ belong to the same equivalence class $j_z^k S$ if and only if the images of the k -tangent morphisms

$$T^k i_S : T^k S \rightarrow T^k Z, \quad T^k i_{S'} : T^k S' \rightarrow T^k Z$$

coincide with each other. The set

$$J_n^k Z = \bigcup_{z \in Z} j_z^k S$$

of k -order jets of submanifolds is a finite-dimensional real smooth manifold, called the k -order jet manifold of submanifolds. For the sake of convenience, we put $J_n^0 Z = Z$.

If $k > 0$, let $Y \rightarrow X$ be an m -dimensional fibre bundle over an n -dimensional base X and $J^k Y$ the k -order jet manifold of sections of $Y \rightarrow X$. Given an imbedding $\Phi : Y \rightarrow Z$, there is the natural injection

$$J^k \Phi : J^k Y \rightarrow J_n^k Z, \quad j_x^k s \rightarrow [\Phi \circ s]_{\Phi(s(x))}^k, \quad (1)$$

where s are sections of $Y \rightarrow X$. This injection defines a chart on $J_n^k Z$. These charts provide a manifold atlas of $J_n^k Z$.

Let us restrict our consideration to first order jets of submanifolds. There is obvious one-to-one correspondence

$$\lambda_{(1)} : j_z^1 S \rightarrow V_{j_z^1 S} \subset T_z Z \quad (2)$$

between the jets $j_z^1 S$ at a point $z \in Z$ and the n -dimensional vector subspaces of the tangent space $T_z Z$ of Z at z . It follows that $J_n^1 Z$ is a fibre bundle

$$\rho : J_n^1 Z \rightarrow Z \quad (3)$$

with the structure group $GL(n, m - n; \mathbb{R})$ of linear transformations of the vector space \mathbb{R}^m which preserve its subspace \mathbb{R}^n . The typical fibre of the fibre bundle (3) is the Grassmann manifold

$$\mathfrak{G}(n, m - n; \mathbb{R}) = GL(m; \mathbb{R}) / GL(n, m - n; \mathbb{R}).$$

This fibre bundle possesses the following coordinate atlas.

Let $\{(U; z^A)\}$ be a coordinate atlas of Z . Though $J_n^0 Z = Z$, let us provide $J_n^0 Z$ with an atlas where every chart $(U; z^A)$ on a domain $U \subset Z$ is replaced with the

$$\binom{m}{n} = \frac{m!}{n!(m-n)!}$$

charts on the same domain U which correspond to different partitions of the collection $(z^1 \cdots z^A)$ in the collections of n and $m-n$ coordinates

$$(U; x^\lambda, y^i), \quad \lambda = 1, \dots, n, \quad i = 1, \dots, m-n. \quad (4)$$

The transition functions between the coordinate charts (4) of $J_n^0 Z$ associated with a coordinate chart (U, z^A) of Z are reduced to exchange between coordinates x^λ and y^i . Transition functions between arbitrary coordinate charts of the manifold $J_n^0 Z$ take the form

$$x'^\lambda = x'^\lambda(x^\mu, y^k), \quad y'^i = y'^i(x^\mu, y^k). \quad (5)$$

Given the coordinate atlas (4) – (5) of a manifold $J_n^0 Z$, the first order jet manifold $J_n^1 Z$ is endowed with an atlas of adapted coordinates

$$(\rho^{-1}(U) = U \times \mathbb{R}^{(m-n)n}; x^\lambda, y^i, y_\lambda^i), \quad (6)$$

possessing transition functions

$$y_\lambda'^i = \left(\frac{\partial y'^i}{\partial y^j} y_\alpha^j + \frac{\partial y'^i}{\partial x^\alpha} \right) \left(\frac{\partial x^\alpha}{\partial y'^k} y_\lambda'^k + \frac{\partial x^\alpha}{\partial x'^\lambda} \right). \quad (7)$$

3 Relativistic mechanics

As was mentioned above, a velocity space of relativistic mechanics is the first order jet manifold $J_1^1 Q$ of one-dimensional submanifolds of a configuration space Q [3, 9, 11].

Given an m -dimensional manifold Q coordinated by (q^λ) , let us consider the jet manifold $J_1^1 Q$ of its one-dimensional submanifolds. Let us provide $Q = J_1^0 Q$ with the coordinates (4):

$$(U; x^0 = q^0, y^i = q^i) = (U; q^\lambda). \quad (8)$$

Then the jet manifold

$$\rho : J_1^1 Q \rightarrow Q$$

is endowed with coordinates (6):

$$(\rho^{-1}(U); q^0, q^i, q_0^i) \quad (9)$$

possessing transition functions (5), (7) which read

$$q'^0 = q'^0(q^0, q^k), \quad q'^0 = q'^0(q^0, q^k), \quad (10)$$

$$q'_0 = \left(\frac{\partial q'^i}{\partial q^j} q_0^j + \frac{\partial q'^i}{\partial q^0} \right) \left(\frac{\partial q'^0}{\partial q^j} q_0^j + \frac{\partial q'^0}{\partial q^0} \right)^{-1}. \quad (11)$$

A glance at the transformation law (11) shows that $J_1^1 Q \rightarrow Q$ is a fibre bundle in projective spaces.

Example 1. Let $Q = M^4 = \mathbb{R}^4$ be a Minkowski space whose Cartesian coordinates (q^λ) , $\lambda = 0, 1, 2, 3$, are subject to the Lorentz transformations (10):

$$q'^0 = q^0 \text{ch} \alpha - q^1 \text{sh} \alpha, \quad q'^1 = -q^0 \text{sh} \alpha + q^1 \text{ch} \alpha, \quad q'^{2,3} = q^{2,3}. \quad (12)$$

Then q'^i (11) are exactly the Lorentz transformations

$$q'_0 = \frac{q_0^1 \text{ch} \alpha - \text{sh} \alpha}{-q_0^1 \text{sh} \alpha + \text{ch} \alpha} \quad q_0'^{2,3} = \frac{q_0^{2,3}}{-q_0^1 \text{sh} \alpha + \text{ch} \alpha}$$

of three-velocities in relativistic mechanics [9, 11].

In view of Example 1, one can think of the velocity space $J_1^1 Q$ of relativistic mechanics as being a space of three-velocities. For the sake of convenience, we agree to call $J_1^1 Q$ the three-velocity space and its coordinate transformations (10) – (11) the relativistic transformations, though a dimension of Q need not equal $3 + 1$.

4 Lagrangian relativistic mechanics

Given the coordinate chart (9) of $J_1^1 Q$, one can regard $\rho^{-1}(U) \subset J_1^1 Q$ as the first order jet manifold $J^1 U$ of sections of the fibre bundle

$$\pi : U \ni (q^0, q^i) \rightarrow (q^0) \in \pi(U) \subset \mathbb{R}. \quad (13)$$

Then three-velocities $(q_0^i) \in \rho^{-1}(U)$ of a relativistic system on U can be treated as absolute velocities of a local non-relativistic system on the configuration space U (13). However, this treatment is broken under the relativistic transformations $q_0^i \rightarrow q_0'^i$ (10) since they are not affine. One can develop first order Lagrangian formalism with a Lagrangian

$$L = \mathcal{L} dq^0 \in \mathcal{O}^{0,1}(\rho^{-1}(U))$$

on a coordinate chart $\rho^{-1}(U)$, but this Lagrangian fails to be globally defined on $J_1^1 Q$ (see Remark 3 below). The graded differential algebra $\mathcal{O}^*(\rho^{-1}(U))$ of exterior forms on $\rho^{-1}(U)$ is generated by horizontal forms dq^0 and contact forms $dq^i - q_0^i dq^0$. Coordinate transformations (10) preserve the ideal of contact forms, but horizontal forms are not transformed into horizontal forms, unless coordinate transition functions q^0 (10) are independent of coordinates q'^i .

In order to overcome this difficulty, let us consider a trivial fibre bundle

$$Q_R = \mathbb{R} \times Q \rightarrow \mathbb{R}, \quad (14)$$

whose base \mathbb{R} is endowed with a Cartesian coordinate τ [5]. This fibre bundle is provided with an atlas of coordinate charts

$$(\mathbb{R} \times U; \tau, q^\lambda), \quad (15)$$

where $(U; q^0, q^i)$ are the coordinate charts (8) of the manifold $J_1^0 Q$. The coordinate charts (15) possess transition functions (10). Let $J^1 Q_R$ be the first order jet manifold of the fibre bundle (14). Since the trivialization (14) is fixed, there is the canonical isomorphism of $J^1 Q_R$ to the vertical tangent bundle

$$J^1 Q_R = VQ_R = \mathbb{R} \times TQ \quad (16)$$

of $Q_R \rightarrow \mathbb{R}$ [5, 6].

Given the coordinate atlas (15) of Q_R , the jet manifold $J^1 Q_R$ is endowed with the coordinate charts

$$((\pi^1)^{-1}(\mathbb{R} \times U) = \mathbb{R} \times U \times \mathbb{R}^m; \tau, q^\lambda, q_\tau^\lambda), \quad (17)$$

possessing transition functions

$$q_\tau'^\lambda = \frac{\partial q'^\lambda}{\partial q^\mu} q_\tau^\mu. \quad (18)$$

Relative to the coordinates (17), the isomorphism (16) takes the form

$$(\tau, q^\mu, q_\tau^\mu) \rightarrow (\tau, q^\mu, \dot{q}^\mu = q_\tau^\mu). \quad (19)$$

Example 2. Let $Q = M^4$ be a Minkowski space in Example 1 whose Cartesian coordinates (q^0, q^i) are subject to the Lorentz transformations (12). Then the corresponding transformations (18) take the form

$$q_\tau'^0 = q_\tau^0 \text{ch} \alpha - q_\tau^1 \text{sh} \alpha, \quad q_\tau'^1 = -q_\tau^0 \text{sh} \alpha + q_\tau^1 \text{ch} \alpha, \quad q_\tau'^{2,3} = q_\tau^{2,3}$$

of transformations of four-velocities in relativistic mechanics.

In view of Example 2, we agree to call fibre elements of $J^1 Q_R \rightarrow Q_R$ the four-velocities though the dimension of Q need not equal 4. Due to the canonical isomorphism $q_\tau^\lambda \rightarrow \dot{q}^\lambda$ (16), by four-velocities also are meant the elements of the tangent bundle TQ , which is called the space of four-velocities.

Obviously, the non-zero jet (19) of sections of the fibre bundle (14) defines some jet of one-dimensional subbundles of the manifold $\{\tau\} \times Q$ through a point $(q^0, q^i) \in Q$, but this is not one-to-one correspondence.

Since non-zero elements of J^1Q_R characterize jets of one-dimensional submanifolds of Q , one hopes to describe the dynamics of one-dimensional submanifolds of a manifold Q as that of sections of the fibre bundle (14). For this purpose, let us refine the relation between elements of the jet manifolds J_1^1Q and J^1Q_R .

Let us consider the manifold product $\mathbb{R} \times J_1^1Q$. It is a fibre bundle over Q_R . Given a coordinate atlas (15) of Q_R , this product is endowed with the coordinate charts

$$(U_R \times \rho^{-1}(U) = U_R \times U \times \mathbb{R}^{m-1}; \tau, q^0, q^i, q_0^i), \quad (20)$$

possessing transition functions (10) – (11). Let us assign to an element (τ, q^0, q^i, q_0^i) of the chart (20) the elements $(\tau, q^0, q^i, q_\tau^0, q_\tau^i)$ of the chart (17) whose coordinates obey the relations

$$q_0^i q_\tau^0 = q_\tau^i. \quad (21)$$

These elements make up a one-dimensional vector space. The relations (21) are maintained under coordinate transformations (11) and (18) [4, 5]. Thus, one can associate:

$$(\tau, q^0, q^i, q_0^i) \rightarrow \{(\tau, q^0, q^i, q_\tau^0, q_\tau^i) \mid q_0^i q_\tau^0 = q_\tau^i\}, \quad (22)$$

to each element of the manifold $\mathbb{R} \times J_1^1Q$ a one-dimensional vector space in the jet manifold J^1Q_R . This is a subspace of elements

$$q_\tau^0(\partial_0 + q_0^i \partial_i)$$

of a fibre of the vertical tangent bundle (16) at a point (τ, q^0, q^i) . Conversely, given a non-zero element (19) of J^1Q_R , there is a coordinate chart (17) such that this element defines a unique element of $\mathbb{R} \times J_1^1Q$ by the relations

$$q_0^i = \frac{q_\tau^i}{q_\tau^0}. \quad (23)$$

Thus, we have shown the following. Let (τ, q^λ) further be arbitrary coordinates on the product Q_R (14) and $(\tau, q^\lambda, q_\tau^\lambda)$ the corresponding coordinates on the jet manifold J^1Q_R .

THEOREM 1. (i) Any jet of submanifolds through a point $q \in Q$ defines some (but not unique) jet of sections of the fibre bundle Q_R (14) through a point $\tau \times q$ for any $\tau \in \mathbb{R}$ in accordance with the relations (21).

(ii) Any non-zero element of J^1Q_R defines a unique element of the jet manifold J_1^1Q by means of the relations (23). However, non-zero elements of J^1Q_R can correspond to different jets of submanifolds.

(iii) Two elements $(\tau, q^\lambda, q_\tau^\lambda)$ and $(\tau, q^\lambda, q_\tau'^\lambda)$ of J^1Q_R correspond to the same jet of submanifolds if $q_\tau'^\lambda = r q_\tau^\lambda$, $r \in \mathbb{R} \setminus \{0\}$. \square

In the case of a Minkowski space $Q = M^4$ in Examples 1 and 2, the equalities (21) and (23) are the familiar relations between three- and four-velocities.

Based on Theorem 1, we can develop Lagrangian theory of one-dimensional submanifolds of a manifold Q as that of sections of the fibre bundle Q_R (14). Let

$$L = \mathcal{L}(\tau, q^\lambda, q_\tau^\lambda) d\tau, \quad (24)$$

be a first order Lagrangian on the jet manifold J^1Q_R . The corresponding Lagrange operator reads

$$\delta L = \mathcal{E}_\lambda dq^\lambda \wedge d\tau, \quad \mathcal{E}_\lambda = \partial_\lambda \mathcal{L} - d_\tau \partial_\lambda^\tau \mathcal{L}. \quad (25)$$

It yields the Lagrange equation

$$\mathcal{E}_\lambda = \partial_\lambda \mathcal{L} - d_\tau \partial_\lambda^\tau \mathcal{L} = 0. \quad (26)$$

In accordance with Theorem 1, it seems reasonable to require that, in order to describe jets of one-dimensional submanifolds of Q , the Lagrangian L (24) on J^1Q_R possesses a gauge symmetry given by vector fields $u = \chi(\tau) \partial_\tau$ on Q_R or, equivalently, their vertical part

$$u_V = -\chi q_\tau^\lambda \partial_\lambda, \quad (27)$$

which are generalized vector fields on Q_R [5, 6]. Then the variational derivatives of this Lagrangian obey the Noether identity:

$$q_\tau^\lambda \mathcal{E}_\lambda = 0. \quad (28)$$

We call such a Lagrangian the relativistic Lagrangian.

In order to obtain a generic form of a relativistic Lagrangian L , let us regard the Noether identity (28) as an equation for L . It admits the following solution. Let

$$\frac{1}{2N!} G_{\alpha_1 \dots \alpha_{2N}}(q^\nu) dq^{\alpha_1} \vee \dots \vee dq^{\alpha_{2N}}$$

be a symmetric tensor field on Q such that the function

$$G = G_{\alpha_1 \dots \alpha_{2N}}(q^\nu) \dot{q}^{\alpha_1} \dots \dot{q}^{\alpha_{2N}} \quad (29)$$

is positive:

$$G > 0, \quad (30)$$

everywhere on $TQ \setminus \widehat{0}(Q)$. Let $A = A_\mu(q^\nu) dq^\mu$ be a one-form on Q . Given the pull-back of G and A onto J^1Q_R due to the canonical isomorphism (16), we define a Lagrangian

$$L = (G^{1/2N} + q_\tau^\mu A_\mu) d\tau, \quad G = G_{\alpha_1 \dots \alpha_{2N}} q_\tau^{\alpha_1} \dots q_\tau^{\alpha_{2N}}, \quad (31)$$

on $J^1Q_R \setminus (\mathbb{R} \times \widehat{0}(Q))$ where $\widehat{0}$ is the global zero section of $TQ \rightarrow Q$. The corresponding Lagrange equation reads

$$\mathcal{E}_\lambda = \frac{\partial_\lambda G}{2NG^{1-1/2N}} - d_\tau \left(\frac{\partial_\lambda^\tau G}{2NG^{1-1/2N}} \right) + F_{\lambda\mu} q_\tau^\mu = \quad (32)$$

$$\begin{aligned} E_\beta [\delta_\lambda^\beta - q_\tau^\beta G_{\lambda\nu_2 \dots \nu_{2N}} q_\tau^{\nu_2} \dots q_\tau^{\nu_{2N}} G^{-1}] G^{1/2N-1} &= 0, \\ E_\beta &= \left(\frac{\partial_\beta G_{\mu\alpha_2 \dots \alpha_{2N}}}{2N} - \partial_\mu G_{\beta\alpha_2 \dots \alpha_{2N}} \right) q_\tau^\mu q_\tau^{\alpha_2} \dots q_\tau^{\alpha_{2N}} - \\ &\quad (2N-1) G_{\beta\mu\alpha_3 \dots \alpha_{2N}} q_{\tau\tau}^\mu q_\tau^{\alpha_3} \dots q_\tau^{\alpha_{2N}} + G^{1-1/2N} F_{\beta\mu} q_\tau^\mu, \\ F_{\lambda\mu} &= \partial_\lambda A_\mu - \partial_\mu A_\lambda. \end{aligned} \quad (33)$$

It is readily observed that the variational derivatives \mathcal{E}_λ (32) satisfy the Noether identity (28). Moreover, any relativistic Lagrangian obeying the Noether identity (28) is of type (31).

A glance at the Lagrange equation (32) shows that it holds if

$$E_\beta = \Phi G_{\beta\nu_2 \dots \nu_{2N}} q_\tau^{\nu_2} \dots q_\tau^{\nu_{2N}} G^{-1}, \quad (34)$$

where Φ is some function on J^1Q_R . In particular, we consider the equation

$$E_\beta = 0. \quad (35)$$

Because of the Noether identity (28), the system of equations (32) is underdetermined. To overcome this difficulty, one can complete it with some additional equation. Given the function G (31), let us choose the condition

$$G = 1. \quad (36)$$

Owing to the property (30), the function G (31) possesses a nowhere vanishing differential. Therefore, its level surface W_G defined by the condition (36) is a submanifold of J^1Q_R .

Our choice of the equation (35) and the condition (36) is motivated by the following facts.

LEMMA 2. Any solution of the Lagrange equation (32) living in the submanifold W_G is a solution of the equation (35). \square

Proof. A solution of the Lagrange equation (32) living in the submanifold W_G obeys the system of equations

$$\mathcal{E}_\lambda = 0, \quad G = 1. \quad (37)$$

Therefore, it satisfies the equality

$$d_\tau G = 0. \quad (38)$$

Then a glance at the expression (32) shows that the equations (37) are equivalent to the equations

$$\begin{aligned} E_\lambda &= \left(\frac{\partial_\lambda G_{\mu\alpha_2\ldots\alpha_{2N}}}{2N} - \partial_\mu G_{\lambda\alpha_2\ldots\alpha_{2N}} \right) q_\tau^\mu q_\tau^{\alpha_2} \cdots q_\tau^{\alpha_{2N}} - \\ &\quad (2N-1) G_{\beta\mu\alpha_3\ldots\alpha_{2N}} q_{\tau\tau}^\mu q_\tau^{\alpha_3} \cdots q_\tau^{\alpha_{2N}} + F_{\beta\mu} q_\tau^\mu = 0, \\ G &= G_{\alpha_1\ldots\alpha_{2N}} q_\tau^{\alpha_1} \cdots q_\tau^{\alpha_{2N}} = 1. \end{aligned} \tag{39}$$

QED

LEMMA 3. Solutions of the equation (35) do not leave the submanifold W_G (36). \square

Proof. Since

$$d_\tau G = -\frac{2N}{2N-1} q_\tau^\beta E_\beta,$$

any solution of the equation (35) intersecting the submanifold W_G (36) obeys the equality (38) and, consequently, lives in W_G . **QED**

The system of equations (39) is called the relativistic equation. Its components E_λ (33) are not independent, but obeys the relation

$$q_\tau^\beta E_\beta = -\frac{2N-1}{2N} d_\tau G = 0, \quad G = 1,$$

similar to the Noether identity (28). The condition (36) is called the relativistic constraint.

Though the equation (32) for sections of a fibre bundle $Q_R \rightarrow \mathbb{R}$ is underdetermined, it is determined if, given a coordinate chart $(U; q^0, q^i)$ (8) of Q and the corresponding coordinate chart (15) of Q_R , we rewrite it in the terms of three-velocities q_0^i (23) as an equation for sections of a fibre bundle $U \rightarrow \pi(U)$ (13).

Let us denote

$$\overline{G}(q^\lambda, q_0^i) = (q_\tau^0)^{-2N} G(q^\lambda, q_\tau^\lambda), \quad q_\tau^0 \neq 0. \tag{40}$$

Then we have

$$\mathcal{E}_i = q_\tau^0 \left[\frac{\partial_i \overline{G}}{2N \overline{G}^{1-1/2N}} - (q_\tau^0)^{-1} d_\tau \left(\frac{\partial_i^0 \overline{G}}{2N \overline{G}^{1-1/2N}} \right) + F_{ij} q_0^j + F_{i0} \right].$$

Let us consider a solution $\{s^\lambda(\tau)\}$ of the equation (32) such that $\partial_\tau s^0$ does not vanish and there exists an inverse function $\tau(q^0)$. Then this solution can be represented by sections

$$s^i(\tau) = (\overline{s}^i \circ s^0)(\tau) \tag{41}$$

of the composite bundle

$$\mathbb{R} \times U \rightarrow \mathbb{R} \times \pi(U) \rightarrow \mathbb{R}$$

where $\bar{s}^i(q^0) = s^i(\tau(q^0))$ are sections of $U \rightarrow \pi(U)$ and $s^0(\tau)$ are sections of $\mathbb{R} \times \pi(U) \rightarrow \mathbb{R}$. Restricted to such solutions, the equation (32) is equivalent to the equation

$$\begin{aligned}\bar{\mathcal{E}}_i &= \frac{\partial_i \bar{G}}{2N\bar{G}^{1-1/2N}} - d_0 \left(\frac{\partial_i^0 \bar{G}}{2N\bar{G}^{1-1/2N}} \right) + \\ F_{ij}q_0^j + F_{i0} &= 0, \\ \bar{\mathcal{E}}_0 &= -q_0^i \bar{\mathcal{E}}_i.\end{aligned}\tag{42}$$

for sections $\bar{s}^i(q^0)$ of a fibre bundle $U \rightarrow \pi(U)$.

It is readily observed that the equation (42) is the Lagrange equation of the Lagrangian

$$\bar{L} = (\bar{G}^{1/2N} + q_0^i A_i + A_0) dq^0\tag{43}$$

on the jet manifold $J^1 U$ of a fibre bundle $U \rightarrow \pi(U)$.

Remark 3. Both the equation (42) and the Lagrangian (43) are defined only on a coordinate chart (8) of Q since they are not maintained by transition functions (10) – (11).

A solution $\bar{s}^i(q^0)$ of the equation (42) defines a solution $s^\lambda(\tau)$ (41) of the equation (32) up to an arbitrary function $s^0(\tau)$. The relativistic constraint (36) enables one to overcome this ambiguity as follows.

Let us assume that, restricted to the coordinate chart $(U; q^0, q^i)$ (8) of Q , the relativistic constraint (36) has no solution $q_\tau^0 = 0$. Then it is brought into the form

$$(q_\tau^0)^{2N} \bar{G}(q^\lambda, q_0^i) = 1,\tag{44}$$

where \bar{G} is the function (40). With the condition (44), every three-velocity (q_0^i) defines a unique pair of four-velocities

$$q_\tau^0 = \pm (\bar{G}(q^\lambda, q_0^i))^{1/2N}, \quad q_\tau^i = q_\tau^0 q_0^i.\tag{45}$$

Accordingly, any solution $\bar{s}^i(q^0)$ of the equation (42) leads to solutions

$$\tau(q^0) = \pm \int (\bar{G}(q^0, \bar{s}^i(q^0), \partial_0 \bar{s}^i(q_0))^{-1/2N} dq^0, \quad s^i(\tau) = s^0(\tau) (\partial_i \bar{s}^i)(s^0(\tau))$$

of the equation (37) and, equivalently, the relativistic equation (39).

Example 4. Let $Q = M^4$ be a Minkowski space provided with the Minkowski metric $\eta_{\mu\nu}$ of signature $(+, ---)$. This is the case of Special Relativity. Let $\mathcal{A}_\lambda dq^\lambda$ be a one-form on Q . Then

$$L = [m(\eta_{\mu\nu} q_\tau^\mu q_\tau^\nu)^{1/2} + e \mathcal{A}_\mu q_\tau^\mu] d\tau, \quad m, e \in \mathbb{R},\tag{46}$$

is a relativistic Lagrangian on $J^1 Q_R$ which satisfies the Noether identity (28). The corresponding relativistic equation (39) reads

$$m \eta_{\mu\nu} q_{\tau\tau}^\nu - e F_{\mu\nu} q_\tau^\nu = 0,\tag{47}$$

$$\eta_{\mu\nu} q_\tau^\mu q_\tau^\nu = 1.\tag{48}$$

This describes a relativistic massive charge in the presence of an electromagnetic field \mathcal{A} . It follows from the relativistic constraint (48) that $(q_\tau^0)^2 \geq 1$. Therefore, passing to three-velocities, we obtain the Lagrangian (43):

$$\bar{L} = \left[m(1 - \sum_i (q_0^i)^2)^{1/2} + e(\mathcal{A}_i q_0^i + \mathcal{A}_0) \right] dq^0,$$

and the Lagrange equation (42):

$$d_0 \left(\frac{mq_0^i}{(1 - \sum_i (q_0^i)^2)^{1/2}} \right) + e(F_{ij}q_0^j + F_{i0}) = 0.$$

Example 5. Let $Q = \mathbb{R}^4$ be an Euclidean space provided with the Euclidean metric ϵ . This is the case of Euclidean Special Relativity. Let $\mathcal{A}_\lambda dq^\lambda$ be a one-form on Q . Then

$$L = [(\epsilon_{\mu\nu} q_\tau^\mu q_\tau^\nu)^{1/2} + \mathcal{A}_\mu q_\tau^\mu] d\tau$$

is a relativistic Lagrangian on $J^1 Q_R$ which satisfies the Noether identity (28). The corresponding relativistic equation (39) reads

$$m\epsilon_{\mu\nu} q_{\tau\tau}^\nu - eF_{\mu\nu} q_\tau^\nu = 0, \tag{49}$$

$$\epsilon_{\mu\nu} q_\tau^\mu q_\tau^\nu = 1. \tag{50}$$

It follows from the relativistic constraint (50) that $0 \leq (q_\tau^0)^2 \leq 1$. Passing to three-velocities, one therefore meets a problem.

5 Relativistic geodesic equations

A glance at the relativistic Lagrangian (31) shows that, because of the gauge symmetry (27), this Lagrangian is independent of τ and, therefore, it describes an autonomous mechanical system. Accordingly, the relativistic equation (39) on Q_R is conservative and, therefore, it is equivalent to an autonomous second order equation on Q whose solutions are parameterized by the coordinate τ on a base \mathbb{R} of Q_R . Given holonomic coordinates $(q^\lambda, \dot{q}^\lambda, \ddot{q}^\lambda)$ of the second tangent bundle $T^2 Q$, this autonomous second order equation (called the autonomous relativistic equation) reads

$$\begin{aligned} & \left(\frac{\partial_\lambda G_{\mu\alpha_2 \dots \alpha_{2N}}}{2N} - \partial_\mu G_{\lambda\alpha_2 \dots \alpha_{2N}} \right) \dot{q}^\mu \dot{q}^{\alpha_2} \dots \dot{q}^{\alpha_{2N}} - \\ & (2N-1)G_{\beta\mu\alpha_3 \dots \alpha_{2N}} \ddot{q}^\mu \dot{q}^{\alpha_3} \dots \dot{q}^{\alpha_{2N}} + F_{\beta\mu} \dot{q}^\mu = 0, \\ & G = G_{\alpha_1 \dots \alpha_{2N}} \dot{q}^{\alpha_1} \dots \dot{q}^{\alpha_{2N}} = 1. \end{aligned} \tag{51}$$

Due to the canonical isomorphism $q_\tau^\lambda \rightarrow \dot{q}^\lambda$ (16), the tangent bundle TQ is regarded as a space of four-velocities.

Generalizing Example 4, let us investigate relativistic mechanics on a four-dimensional pseudo-Riemannian manifold $Q = X$, coordinated by (x^λ) and provided with a pseudo-Riemannian metric g of signature $(+, - - -)$. We agree to call X a world manifold. Let $A = A_\lambda dx^\lambda$ be a one-form on X . Let us consider the relativistic Lagrangian (31):

$$L = [(g_{\alpha\beta}x_\tau^\alpha x_\tau^\beta)^{1/2} + A_\mu x_\tau^\mu]d\tau,$$

and the relativistic constraint (36):

$$g_{\alpha\beta}x_\tau^\alpha x_\tau^\beta = 1.$$

The corresponding autonomous relativistic equation (39) on X takes the form

$$\ddot{x}^\lambda - \{\mu^\lambda{}_\nu\}\dot{x}^\mu\dot{x}^\nu - g^{\lambda\beta}F_{\beta\nu}\dot{x}^\nu = 0, \quad (52)$$

$$g = g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 1, \quad (53)$$

where $\{\mu^\lambda{}_\nu\}$ is the Levi-Civita connection. A glance at the equality (52) shows that it is a geodesic equation on TX with respect to an affine connection

$$K_\mu^\lambda = \{\mu^\lambda{}_\nu\}\dot{x}^\nu + g^{\lambda\nu}F_{\nu\mu}. \quad (54)$$

on TX .

A particular form of this connection follows from the fact that the geodesic equation (52) is derived from a Lagrange equation, i.e., we are in the case of Lagrangian relativistic mechanics. In a general setting, relativistic mechanics on a pseudo-Riemannian manifold (X, g) can be formulated as follows.

The geodesic equation

$$\ddot{x}^\mu = K_\lambda^\mu(x^\nu, \dot{x}^\nu)\dot{x}^\lambda, \quad (55)$$

on the tangent bundle TX with respect to a connection

$$K = dx^\lambda \otimes (\partial_\lambda + K_\lambda^\mu \dot{\partial}_\mu) \quad (56)$$

on $TX \rightarrow X$ is called a relativistic geodesic equation if a geodesic vector field of K lives in the subbundle of hyperboloids

$$W_g = \{\dot{x}^\lambda \in TX \mid g_{\lambda\mu}\dot{x}^\lambda\dot{x}^\mu = 1\} \subset TX \quad (57)$$

defined by the relativistic constraint (53).

One can show that the equation (55) is a relativistic geodesic equation if the condition

$$(\partial_\lambda g_{\mu\nu}\dot{x}^\mu + 2g_{\mu\nu}K_\lambda^\mu)\dot{x}^\lambda\dot{x}^\nu = 0 \quad (58)$$

holds.

Obviously, the connection (54) fulfils the condition (58). Any metric connection, e.g., the Levi-Civita connection $\{\lambda^\mu{}_\nu\}$ on TX satisfies the condition (58).

Given a Levi-Civita connection $\{\lambda^\mu{}_\nu\}$, any connection K on $TX \rightarrow X$ can be written as

$$K^\mu_\lambda = \{\lambda^\mu{}_\nu\} \dot{x}^\nu + \sigma^\mu_\lambda(x^\lambda, \dot{x}^\lambda), \quad (59)$$

where

$$\sigma = \sigma^\mu_\lambda dx^\lambda \otimes \dot{\partial}_\lambda \quad (60)$$

is some soldering form on TX . Then the condition (58) takes the form

$$g_{\mu\nu} \sigma^\mu_\lambda \dot{x}^\lambda \dot{x}^\nu = 0. \quad (61)$$

With the decomposition (59), one can think of the relativistic geodesic equation (55):

$$\ddot{x}^\mu = \{\lambda^\mu{}_\nu\} \dot{x}^\nu \dot{x}^\lambda + \sigma^\mu_\lambda(x^\lambda, \dot{x}^\lambda) \dot{x}^\lambda, \quad (62)$$

as describing a relativistic particle in the presence of a gravitational field g and a non-gravitational external force σ .

6 Hamiltonian relativistic mechanics

We are in the case of relativistic mechanics on a pseudo-Riemannian world manifold (X, g) . Given the coordinate chart (13) of its configuration space X , the homogeneous Legendre bundle corresponding to the local non-relativistic system on U is the cotangent bundle T^*U of U . This fact motivate us to think of the cotangent bundle T^*X as being the phase space of relativistic mechanics on X . It is provided with the canonical symplectic form

$$\Omega = dp_\lambda \wedge dx^\lambda \quad (63)$$

and the corresponding Poisson bracket $\{\cdot, \cdot\}$.

A relativistic Hamiltonian is defined as follows [9, 10, 11]. Let H be a smooth real function on T^*X such that the morphism

$$\widetilde{H} : T^*X \rightarrow TX, \quad \dot{x}^\mu \circ \widetilde{H} = \partial^\mu H, \quad (64)$$

is a bundle isomorphism. Then the inverse image

$$N = \widetilde{H}^{-1}(W_g)$$

of the subbundle of hyperboloids W_g (57) is a one-codimensional (consequently, coisotropic) closed imbedded subbundle N of T^*X given by the condition

$$H_T = g_{\mu\nu} \partial^\mu H \partial^\nu H - 1 = 0. \quad (65)$$

We say that H is a relativistic Hamiltonian if the Poisson bracket $\{H, H_T\}$ vanishes on N . This means that the Hamiltonian vector field

$$\gamma = \partial^\lambda H \partial_\lambda - \partial_\lambda H \partial^\lambda \quad (66)$$

of H preserves the constraint N and, restricted to N , it obeys the equation

$$\gamma \rfloor \Omega_N + i_N^* dH = 0, \quad (67)$$

which is the Hamilton equation of a Dirac constrained system on N with a Hamiltonian H [3].

The morphism (64) sends the vector field γ (66) onto the vector field

$$\gamma_T = \dot{x}^\lambda \partial_\lambda + (\partial^\mu H \partial^\lambda \partial_\mu H - \partial_\mu H \partial^\lambda \partial^\mu H) \dot{\partial}_\lambda$$

on TX . This vector field defines the autonomous second order dynamic equation

$$\ddot{x}^\lambda = \partial^\mu H \partial^\lambda \partial_\mu H - \partial_\mu H \partial^\lambda \partial^\mu H \quad (68)$$

on X which preserves the subbundle of hyperboloids (57), i.e., it is the autonomous relativistic equation (51).

Example 6. The following is a basic example of relativistic Hamiltonian mechanics. Given a one-form $A = A_\mu dq^\mu$ on X , let us put

$$H = g^{\mu\nu} (p_\mu - A_\mu)(p_\nu - A_\nu). \quad (69)$$

Then $H_T = 2H - 1$ and, hence, $\{H, H_T\} = 0$. The constraint $H_T = 0$ (65) defines a one-codimensional closed imbedded subbundle N of T^*X . The Hamilton equation (67) takes the form $\gamma \rfloor \Omega_N = 0$. Its solution (66) reads

$$\begin{aligned} \dot{x}^\alpha &= g^{\alpha\nu} (p_\nu - A_\nu), \\ \dot{p}_\alpha &= -\frac{1}{2} \partial_\alpha g^{\mu\nu} (p_\mu - A_\mu)(p_\nu - A_\nu) + g^{\mu\nu} (p_\mu - A_\mu) \partial_\alpha A_\nu. \end{aligned}$$

The corresponding autonomous second order dynamic equation (68) on X is

$$\begin{aligned} \ddot{x}^\lambda - \{\mu^\lambda{}_\nu\} \dot{x}^\mu \dot{x}^\nu - g^{\lambda\nu} F_{\nu\mu} \dot{x}^\mu &= 0, \\ \{\mu^\lambda{}_\nu\} &= -\frac{1}{2} g^{\lambda\beta} (\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}), \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned} \quad (70)$$

It is a relativistic geodesic equation with respect to the affine connection (54).

Since the equation (70) coincides with the generic Lagrange equation (52) on a world manifold X , one can think of H (69) as being a generic Hamiltonian of relativistic mechanics on X .

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